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The bed configuration of straight sand-bed channels when flow is nearly critical

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The diagonal pattern of bed form which appears in sand-bed channels when the channel width to depth ratio is great and the flow is nearly critical is discussed from the theoretical viewpoint by using the method of characteristics. Some photographs illustrate the phenomenon.

1. Introduction

The diagonal sand waves occur in straight sand-bed channels with smooth walls, when the flow is nearly critical (or the Froude number is near unity) with a certain depth to width ratio. This phenomenon has been observed previously by Vanoni & Brooks (1957), Shen (1961) and Guy, Simons & Richardson (1966) in their studies of resistance to flow and bed-material discharge in an 8 foot wide flume at Colorado State University during the period 1956–1958. According to Shen, this kind of diagonal pattern is probably due to water-surface fluctuations.

From further observations by the authors, the diagonal sand waves on a sand bed are associated with the water-surface undulation which is a type of disturbance that occurs when flow changes from supercritical to subcritical or vice versa. Flow bounded within the diagonal disturbances is essentially continuous, whereas a discontinuity exists for flow across the disturbances, see figure 1. The discontinuity in a flow field is usually determined by the method of characteristics which is used most often for problems in the field of gas dynamics (Owczarek 1964).

2. Theoretical considerations

In a straight alluvial channel with a large width/depth ratio, the vertical motion is neglected and the equations of motion in the longitudinal and transverse directions are

$$\frac{1}{g}\left(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+W\frac{\partial U}{\partial z}\right)=S-\frac{\partial h}{\partial x}-\frac{\tau_0}{\gamma h},\tag{1}$$

$$\frac{1}{g}\left(\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial x}+W\frac{\partial W}{\partial z}\right)=-\frac{\partial h}{\partial z}-\frac{W}{U}\frac{\tau_0}{\gamma h},\tag{2}$$

where U and W are average velocities per unit width in the longitudinal and transverse directions, S is the channel slope, h the local depth of flow, and $\tau_0 = \gamma h S$ (shear stress at channel bottom).

The continuity equation for water discharge is

$$\frac{\partial}{\partial x}(Uh)+\frac{\partial}{\partial z}(Wh) = -\frac{\partial h}{\partial t}. \tag{3}$$

The continuity for sediment transport requires

$$\frac{\partial q_1}{\partial x}+\frac{\partial q_3}{\partial z} = -(1-\lambda)\frac{\partial h_0}{\partial t},$$

where q_1 and q_3 are the discharges of sands per unit width in the longitudinal and transverse directions, λ is the porosity of sands, and h_0 is the channel-bed elevation.

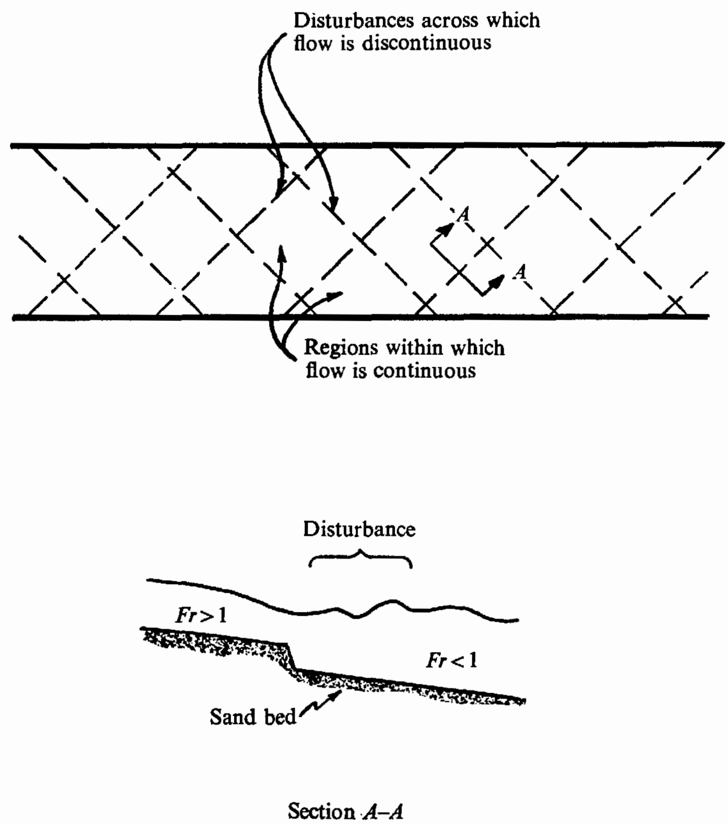

FIGURE 1. Schematic drawing showing diagonal lines in shallow channel flow with Froude number near unity.

The discharge of sands in a sand-bed channel depends on many variables. Colby (1964) has found that in streams where differences in depth, water temperature, and size of bed sand are not excessively large, the discharge of sands per unit width is proportional to the average velocity. If we assume that the discharges of sands are proportional to the velocity components, i.e.

$$U/W = q_1/q_3,$$

then

$$\frac{\partial q_1}{\partial x}+\frac{\partial}{\partial z}\left(\frac{W}{U}q_1\right) = -(1-\lambda)\frac{\partial h_0}{\partial t}. \tag{4}$$

The movement of diagonal sand waves has been observed to be very slow, hence the unsteady terms in the above equations can be neglected. The above four equations, together with the equations for total differentials can be written as

$$U\frac{\partial U}{\partial x}+W\frac{\partial U}{\partial z} = F_1(h,x), \tag{5}$$

$$U\frac{\partial W}{\partial x}+W\frac{\partial W}{\partial z} = F_2(h,z), \tag{6}$$

$$h\frac{\partial U}{\partial x}+h\frac{\partial W}{\partial z}+U\frac{\partial h}{\partial x}+W\frac{\partial h}{\partial z} = 0, \tag{7}$$

$$\frac{Wq_1}{U^2}\frac{\partial U}{\partial z}-\frac{q_1}{U}\frac{\partial W}{\partial z}-\frac{\partial q_1}{\partial x}-\frac{W}{U}\frac{\partial q_1}{\partial z} = 0, \tag{8}$$

$$dx\frac{\partial U}{\partial x}+dz\frac{\partial U}{\partial z} = dU, \tag{9}$$

$$dx\frac{\partial W}{\partial x}+dz\frac{\partial W}{\partial z} = dW, \tag{10}$$

$$dx\frac{\partial h}{\partial x}+dz\frac{\partial h}{\partial z} = dh, \tag{11}$$

$$dx\frac{\partial q_1}{\partial x}+dz\frac{\partial q_1}{\partial z} = dq_1. \tag{12}$$

This system of simultaneous equations for the partial derivatives $\partial U/\partial x$, $\partial U/dz$, $\partial W/\partial x$, $\partial W/\partial z$, $\partial h/\partial x$, $\partial h/\partial z$, $\partial q_1/\partial x$ and $\partial q_1/\partial z$ has independent variables x and z, and dependent variables U, W, h and q_1. The coefficients of the partial derivatives in (5) to (8) are functions of dependent variables only. This system of equations is called quasi-linear because each equation is linear with respect to the derivatives of the highest (in this case, first) order. Therefore, these equations can be analyzed by the method of characteristics.

Using Cramer's rule, the derivative $\partial U/\partial x$ can be determined from the quotient of two determinants

$$\partial U/\partial x = k_1/N,$$

where

$$k_1 = \begin{vmatrix} F_1 & W & 0 & 0 & g & 0 & 0 & 0 \\ F_2 & 0 & U & W & 0 & g & 0 & 0 \\ 0 & 0 & 0 & h & U & W & 0 & 0 \\ 0 & \frac{Wq_1}{U^2} & 0 & -\frac{q_1}{U} & 0 & 0 & -1 & -\frac{W}{U} \\ dU & dz & 0 & 0 & 0 & 0 & 0 & 0 \\ dW & 0 & dx & dz & 0 & 0 & 0 & 0 \\ dh & 0 & 0 & 0 & dx & dz & 0 & 0 \\ dq_1 & 0 & 0 & 0 & 0 & 0 & dx & dz \end{vmatrix},$$

and

$$N = \begin{vmatrix} U & W & 0 & 0 & g & 0 & 0 & 0 \\ 0 & 0 & U & W & 0 & g & 0 & 0 \\ h & 0 & 0 & h & U & W & 0 & 0 \\ 0 & \dfrac{Wq_1}{U^2} & 0 & -\dfrac{q_1}{U} & 0 & 0 & -1 & -\dfrac{W}{U} \\ dx & dz & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & dx & dz & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & dx & dz & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & dx & dz \end{vmatrix}$$

Similarly, other derivatives are

$$\frac{\partial U}{\partial z} = \frac{k_2}{N}, \quad \frac{\partial W}{\partial x} = \frac{k_3}{N}, \quad \frac{\partial W}{\partial z} = \frac{k_4}{N}, \quad \frac{\partial h}{\partial x} = \frac{k_5}{N}$$

$$\frac{\partial h}{\partial z} = \frac{k_6}{N}, \quad \frac{\partial q_1}{\partial x} = \frac{k_7}{N}, \quad \text{and} \quad \frac{\partial q_1}{\partial z} = \frac{k_8}{N},$$

where $k_2, k_3, \ldots, k_8$ are appropriate determinants.

The necessary condition for the partial derivatives to be indeterminate, or there would be a discontinuity in the flow field, is that the determinant $N = 0$. The directions in the (x, z) plane in which the determinant $N = 0$ are called characteristic directions and curves along which $N = 0$ are called characteristic curves. If the flow under consideration permits the existence of discontinuities in the form of water surface undulations, their paths can only be represented by the characteristic curves. Hence, the characteristic curves may represent the diagonal paths of disturbances.

Letting $N = 0$, we obtain

$$(W\,dx - U\,dz)^2\,[(W\,dx - U\,dz)^2 - gh(dx)^2] = 0. \tag{13}$$

The four roots of the above equation are

$$\frac{dx}{dz} = \frac{U}{W}, \quad \frac{dx}{dz} = \frac{U}{W}, \quad \frac{dx}{dz} = \frac{U}{W + \sqrt{(gh)}} \quad \text{and} \quad \frac{dx}{dz} = \frac{U}{W - \sqrt{(gh)}}.$$

They are independent of the sediment discharge. The first two roots are stream lines, and the characteristic direction represented by the last two roots is of principal interest to us. These roots can be written as

$$\frac{dx}{dz} = \frac{U}{W \pm \sqrt{(gh)}}. \tag{14}$$

In this equation, if we assume that W is small and hence negligible, then

$$dx/dz = \pm\, U/\sqrt{(gh)}. \tag{15}$$

Since water surface undulations occur when the flow is nearly critical, or $U \doteq \sqrt{(gh)}$, then

$$dx/dz \doteq \pm 1. \tag{16}$$

Equation (16) indicates that the disturbances occur on lines approximately 45° from the flow direction. Thus, we have verified that the sand-wave patterns associated with the accompanying water surface undulations are diagonal. Figure 2, plates 1 and 2, shows the diagonal bed form of channel flows when the water had been shut off, as observed by Guy, Simons & Richardson in an 8 foot wide laboratory flume. The Froude numbers ($F_r = U/\sqrt{(gh)}$) and width to depth ratios for all runs are also listed.

3. Conclusions

The diagonal bed form usually occurs in alluvial channels with a large width to depth ratio and with the flow nearly critical (or the Froude number near unity). The diagonal bed form is associated with the water-surface undulation which is a disturbance across which the flow changes from supercritical to subcritical or vice versa. It has been verified in this paper that the disturbance or the sand wave occurs on lines approximately 45° from the flow direction.

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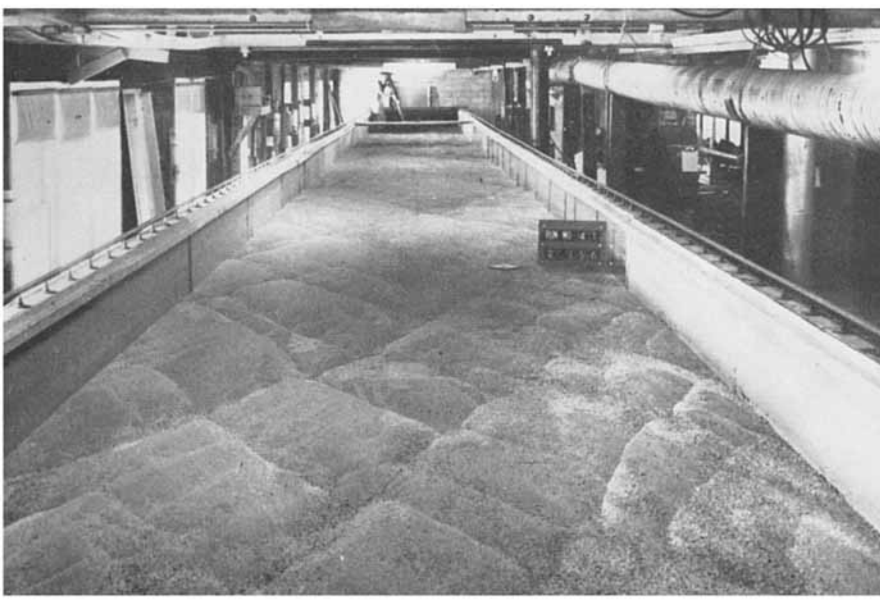
(*a*)

(*b*)

FIGURE 2(*a*), (*b*). For legend see plate 2.



(c)

FIGURE 2. Diagonal bed patterns in a laboratory flume with large width to depth ratios and with the flow nearly critical. (*a*) Froude number = 0·92, width to depth ratio = 24. (*b*) Froude number = 0·83, width to depth ratio = 28·5. (*c*) Froude number = 1·12, width to depth ratio = 18.